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Self-dual gravity is completely integrable

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Abstract

We discover a multi-Hamiltonian structure of a complex Monge–Ampère equation (CMA) set in a real first-order 2-component form. Therefore, by Magri’s theorem this is a completely integrable system in four real dimensions. We start with Lagrangian and Hamiltonian densities and obtain a symplectic form and the Hamiltonian operator that determines the Dirac bracket. We have calculated all point symmetries of the 2-component CMA system and Hamiltonians of the symmetry flows. We have found two new real recursion operators for symmetries which commute with the operator of a symmetry condition on solutions of the CMA system. These operators form two Lax pairs for the 2-component system. The recursion operators, applied to the first Hamiltonian operator, generate infinitely many real Hamiltonian structures. We show how to construct an infinite hierarchy of higher commuting flows together with the corresponding infinite chain of their Hamiltonians.

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1. Introduction

In an earlier paper [1] we presented a complex multi-Hamiltonian structure of Plebański’s second heavenly equation [2], which by Magri’s theorem [3] proves that it is a completely integrable system in four complex dimensions. We expect that Plebański’s first heavenly equation also admits a multi-Hamiltonian structure, since these two equations, governing Ricci-flat metrics with (anti-)self-dual Riemann curvature 2-form, are related by a Legendre transformation of the corresponding heavenly tetrads [2]. However, since both Plebański’s equations are complex, their solutions are potentials of the complex metrics that satisfy Einstein equations in complex four-dimensional spaces. In the case of the complex Monge–Ampère equation (CMA), which governs the (anti-)self-dual gravity in real four-dimensional

spaces with either Euclidean or ultra-hyperbolic signature, we have an additional condition that symplectic, Hamiltonian and recursion operators should all be real. Furthermore, the transformation between the two heavenly equations cannot be applied to transform the second heavenly equation to CMA because the latter equation is real and therefore the multi-Hamiltonian structure of CMA cannot be obtained by transforming the multi-Hamiltonian structure of the second heavenly equation given in [1]. Therefore, in this paper we analyze the complex Monge–Ampère equation independently of our previous work and obtain real recursion operators for symmetries and real multi-Hamiltonian structures of CMA.

In section 2, we start with the complex Monge–Ampère equation in a 2-component first-order evolutionary form with the Lagrangian that is appropriate for Hamiltonian formulation. In section 3, we discover a symplectic structure and Hamiltonian structure of this CMA system. In section 4, we transform the Hamiltonian density and Hamiltonian operator to real variables and introduce a convenient notation needed later to arrive at a compact form of recursion operators and higher Hamiltonian operators. In section 5, we derive a symmetry condition, which determines symmetries of the CMA system, in a 2-component form and real variables, using our new notation. We have calculated all point symmetries of the CMA system and Hamiltonians of the symmetry flows that yield conservation laws for the CMA system. In section 6 we obtain two new real recursion operators for symmetries which commute with an operator of the symmetry condition on solutions of the CMA system. Moreover, these two couples of operators form two Lax pairs for the 2-component system. In section 7 we apply these recursion operators to the first Hamiltonian operator and discover further Hamiltonian structures of the CMA system. Repeating this procedure for the second Hamiltonian operator, we could generate infinitely many Hamiltonian structures of the CMA system. This multi-Hamiltonian structure of the CMA system proves its complete integrability in the sense of Magri and hence complete integrability of the (anti-)self-dual gravity in four real dimensions with either Euclidean or ultra-hyperbolic signature. In section 8, we construct an infinite hierarchy of higher flows and show the way of calculating the corresponding infinite chain of higher Hamiltonians.

2. An evolutionary first-order form of the complex Monge–Ampère equation and its Lagrangian

The four-dimensional hyper-Kähler metrics

$$ds^2 = u_{1\bar{1}} dz^1 d\bar{z}^1 + u_{1\bar{2}} dz^1 d\bar{z}^2 + u_{2\bar{1}} dz^2 d\bar{z}^1 + u_{2\bar{2}} dz^2 d\bar{z}^2 \tag{2.1}$$

satisfy Einstein field equations with either Euclidean or ultra-hyperbolic signature, if the Kähler potential u satisfies the elliptic or hyperbolic complex Monge–Ampère equation

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = \varepsilon \tag{2.2}$$

with $\varepsilon = \pm 1$ respectively [2]. Here u is a real-valued function of the two complex variables z^1, z^2 and their conjugates \bar{z}^1, \bar{z}^2 , the subscripts denoting partial derivatives with respect to these variables. Such metrics are Ricci-flat and have (anti-)self-dual curvature.

In order to discuss the Hamiltonian structure of CMA (2.2), we shall replace the complex conjugate pair of variables z^1, \bar{z}^1 by the real time variable $t = 2\Re z^1$ and the real space variable $x = 2\Im z^1$ and change the notation for the second complex variable $z^2 = w$. Then (2.2) becomes

$$(u_{tt} + u_{xx})u_{w\bar{w}} - u_{tw}u_{t\bar{w}} - u_{xw}u_{x\bar{w}} + i(u_{tw}u_{x\bar{w}} - u_{xw}u_{t\bar{w}}) = \varepsilon. \tag{2.3}$$

Now we can express (2.3) as a pair of first-order nonlinear evolution equations by introducing an auxiliary dependent variable $v = u_t$,

$$\begin{cases} u_t = v \\ v_t = -u_{xx} + \frac{1}{u_{w\bar{w}}}(v_w v_{\bar{w}} + u_{xw} u_{x\bar{w}} + i(v_{\bar{w}} u_{xw} - v_w u_{x\bar{w}}) + \varepsilon), \end{cases} \quad (2.4)$$

so that finally (2.2) is set in a 2-component form. For the sake of brevity we shall henceforth refer to (2.4) as the CMA system.

The Lagrangian density for the original form (2.2) of the complex Monge–Ampère equation was suggested in [4]

$$L = \frac{1}{6}[u_1 u_{\bar{1}} u_{2\bar{2}} + u_2 u_{\bar{2}} u_{1\bar{1}} - u_1 u_{\bar{2}} u_{2\bar{1}} - u_2 u_{\bar{1}} u_{1\bar{2}}] + \varepsilon u, \quad (2.5)$$

but this must be cast into a form suitable for passing to a Hamiltonian. This requires that the form of a Lagrangian should be appropriate for applying Dirac’s theory of constraints [5]. We choose the Lagrangian density for the first-order CMA system (2.4) to be degenerate, that is, linear in the time derivative of unknown u_t and with no v_t :

$$L = \frac{1}{6}\{(u_x^2 - 3v^2)u_{w\bar{w}} + u_w u_{\bar{w}} u_{xx} - u_x(u_w u_{x\bar{w}} + u_{\bar{w}} u_{xw}) + u_t(2i(u_w u_{x\bar{w}} - u_{\bar{w}} u_{xw}) + 6v u_{w\bar{w}})\} + \varepsilon u \quad (2.6)$$

which, after substituting $v = u_t$, coincides with our original Lagrangian (2.5) up to a total divergence.

3. Symplectic and Hamiltonian structures

Since the Lagrangian density (2.6) is linear in u_t and has no v_t , the canonical momenta

$$\begin{aligned} \pi_u &= \frac{\partial L}{\partial u_t} = \frac{i}{3}(u_w u_{x\bar{w}} - u_{\bar{w}} u_{xw}) + v u_{w\bar{w}} \\ \pi_v &= \frac{\partial L}{\partial v_t} = 0 \end{aligned} \quad (3.1)$$

cannot be inverted for the velocities u_t and v_t and so the Lagrangian is degenerate. Therefore, according to Dirac’s theory [5], we impose them as constraints

$$\begin{aligned} \phi_u &= \pi_u + \frac{i}{3}(u_{\bar{w}} u_{xw} - u_w u_{x\bar{w}}) - v u_{w\bar{w}} = 0 \\ \phi_v &= \pi_v = 0 \end{aligned} \quad (3.2)$$

and calculate the Poisson brackets of the constraints (more details of this procedure were given in [1])

$$K_{ik} = [\phi_i(x, w, \bar{w}), \phi_k(x', w', \bar{w}')] \quad (3.3)$$

collecting results in a 2×2 matrix form, where the subscripts run from 1 to 2 with 1 and 2 corresponding to u and v , respectively. This yields the symplectic operator K that is the inverse of the Hamiltonian operator J_0 :

$$K = \begin{pmatrix} (v_{\bar{w}} - i u_{x\bar{w}})D_w + (v_w + i u_{xw})D_{\bar{w}} + v_{w\bar{w}} & -u_{w\bar{w}} \\ u_{w\bar{w}} & 0 \end{pmatrix} \quad (3.4)$$

as an explicitly skew-symmetric local operator. The symplectic 2-form is a volume integral $\Omega = \int_V \omega \, dx \, dw \, d\bar{w}$ of the density

$$\omega = \frac{1}{2} du^i \wedge K_{ij} du^j = \frac{1}{2}(v_{\bar{w}} - i u_{x\bar{w}}) du \wedge du_w + \frac{1}{2}(v_w + i u_{xw}) du \wedge du_{\bar{w}} + u_{w\bar{w}} dv \wedge du \quad (3.5)$$

where $u^1 = u$ and $u^2 = v$. In ω , under the sign of the volume integral, we can neglect all the terms that are either total derivatives or total divergencies due to suitable boundary conditions on the boundary surface of the volume.

For the exterior differential of this 2-form we obtain

$$\begin{aligned} d\omega &= -idu_x \wedge du_w \wedge du_{\bar{w}} \\ &= -(i/3)(D_x(du \wedge du_w \wedge du_{\bar{w}}) + D_w(du_x \wedge du \wedge du_{\bar{w}}) \\ &\quad + D_{\bar{w}}(du_x \wedge du_w \wedge du)) \iff 0, \end{aligned} \tag{3.6}$$

that is, a total divergence which is equivalent to zero, so that the 2-form Ω is closed and hence symplectic. The Hamiltonian operator J_0 is obtained by inverting K in (3.4)

$$J_0 = \begin{pmatrix} 0 & \frac{1}{u_w \bar{w}} \\ -\frac{1}{u_w \bar{w}} & \frac{v_{\bar{w}} - iu_{x\bar{w}}}{2u_w^2 \bar{w}} D_w + D_w \frac{v_{\bar{w}} - iu_{x\bar{w}}}{2u_w^2 \bar{w}} + \frac{v_w + iu_{xw}}{2u_w^2 \bar{w}} D_{\bar{w}} + D_{\bar{w}} \frac{v_w + iu_{xw}}{2u_w^2 \bar{w}} \end{pmatrix} \tag{3.7}$$

which is explicitly skew-symmetric. It satisfies the Jacobi identity due to (3.6).

The Hamiltonian density is

$$H_1 = \pi_u u_t + \pi_v v_t - L$$

with the result

$$H_1 = \frac{1}{6}[(3v^2 - u_x^2)u_w \bar{w} - u_w u_{\bar{w}} u_{xx} + u_x(u_{\bar{w}} u_{xw} + u_w u_{x\bar{w}})] - \varepsilon u. \tag{3.8}$$

The CMA system can now be written in the Hamiltonian form with the Hamiltonian density H_1 defined by (3.8)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}, \tag{3.9}$$

where δ_u and δ_v are Euler–Lagrange operators [6] with respect to u and v applied to the Hamiltonian density H_1 (they correspond to the variational derivatives of the Hamiltonian functional $\int_V H_1 dV$).

4. Transformation to real variables

In the case of CMA, which governs (anti-)self-dual gravity with either Euclidean or ultra-hyperbolic signature, we have an additional condition that all the objects in the theory, in particular a recursion operator, should be real. Therefore, we transform the Hamiltonian density together with the symplectic and Hamiltonian operators to real variables $y = 2\text{Re}w$ and $z = 2\text{Im}w$. The Hamiltonian density in the real variables becomes

$$H_1 = \frac{1}{6}[(3v^2 - u_x^2)\Delta(u) - (u_y^2 + u_z^2)u_{xx} + 2u_x(u_y u_{xy} + u_z u_{xz})] - \varepsilon u$$

where $\Delta(u) = u_{yy} + u_{zz}$, which simplifies after canceling terms that are total derivatives as

$$H_1 = \frac{1}{2}[v^2 \Delta(u) - u_{xx}(u_y^2 + u_z^2)] - \varepsilon u. \tag{4.1}$$

The transformation of the Hamiltonian operator J_0 in (3.7) yields

$$J_0 = \begin{pmatrix} 0 & \frac{1}{a} \\ -\frac{1}{a} & \frac{1}{a^2}(cD_y - bD_z) + (D_y c - D_z b)\frac{1}{a^2} \end{pmatrix}, \tag{4.2}$$

where we introduce the notation

$$a = \Delta(u), \quad b = u_{xy} - v_z, \quad c = v_y + u_{xz}, \quad Q = \frac{b^2 + c^2 + \varepsilon}{a} \tag{4.3}$$

that we will use from now on throughout the paper, with D_y and D_z designating operators of total derivatives with respect to y and z , respectively. The definitions (4.3) imply the relations

$$a_x = b_y + c_z, \quad c_y - b_z = \Delta(v), \tag{4.4}$$

where $\Delta = D_y^2 + D_z^2$ is the two-dimensional Laplace operator.

The symplectic operator (3.4) in the real variables becomes

$$K = J_0^{-1} = \begin{pmatrix} cD_y - bD_z + D_y c - D_z b & -a \\ a & 0 \end{pmatrix} \tag{4.5}$$

in an explicitly skew-symmetric form.

The CMA system (2.4) in the real variables becomes

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} v \\ Q - u_{xx} \end{pmatrix} \tag{4.6}$$

or $u_t = v, v_t = Q - u_{xx}$.

As a consequence of the definitions (4.3) and equations of motion (4.6), we have the following useful relations:

$$a_t = c_y - b_z, \quad b_t = c_x - Q_z, \quad c_t = Q_y - b_x, \tag{4.7}$$

$$\begin{aligned} Q_t &= \frac{2c(Q_y - b_x) + 2b(c_x - Q_z) - (c_y - b_z)Q}{a}, & Q_x &= \frac{2cc_x + 2bb_x}{a} - \frac{a_x}{a}Q \\ Q_y &= \frac{2cc_y + 2bb_y}{a} - \frac{a_y}{a}Q, & Q_z &= \frac{2cc_z + 2bb_z}{a} - \frac{a_z}{a}Q. \end{aligned} \tag{4.8}$$

The four-dimensional hyper-Kähler metrics (2.1) in the real variables in the notation (4.3) become

$$ds^2 = \frac{1}{4} [Q(dt^2 + dx^2) + a(dy^2 + dz^2)] + \frac{1}{2} [c(dt dy + dx dz) - b(dt dz - dx dy)]. \tag{4.9}$$

The metrics (4.9) satisfy Einstein field equations with either Euclidean or ultra-hyperbolic signature, if the 2-component potential (u, v) in the definitions (4.3) of a, b, c and Q satisfies the Hamiltonian CMA system (4.6) with $\varepsilon = +1$ or $\varepsilon = -1$, respectively. These metrics are again Ricci-flat and have (anti-)self-dual curvature.

5. Symmetries and integrals of motion

Now, consider Lie group of transformations of the system (4.6) in the evolutionary form, when only dependent variables are transformed, and let τ be the group parameter. Then Lie equations read

$$u_\tau = \varphi, \quad v_\tau = \psi, \tag{5.1}$$

where $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is a 2-component symmetry characteristic of the system (4.6). The differential compatibility conditions of equations (4.6) and (5.1) in the form $u_{t\tau} - u_{\tau t} = 0$ and $v_{t\tau} - v_{\tau t} = 0$ result in the linear matrix equation

$$\mathcal{A}(\Phi) = 0, \tag{5.2}$$

where \mathcal{A} is the Frechét derivative of the flow (4.6):

$$\mathcal{A} = \begin{pmatrix} D_t & -1 \\ D_x^2 - \frac{2}{a}(cD_z + bD_y)D_x + \frac{Q}{a}\Delta, & D_t - \frac{2}{a}(cD_y - bD_z) \end{pmatrix} \tag{5.3}$$

where the first row of (5.2) yields $\varphi_t = \psi$.

Using the software packages LIEPDE and CRACK by Wolf [7], run under REDUCE 3.8, we have calculated all point symmetries of the CMA system (4.6), a class of solutions of the matrix equation (5.2). We list their generators and 2-component symmetry characteristics [6], the latter being denoted by φ^u, φ^v

$$\begin{aligned}
 X_1 &= t\partial_t + x\partial_x + u\partial_u, & \varphi_1^u &= u - tv - xu_x, & \varphi_1^v &= t(u_{xx} - Q) - xv_x \\
 X_2 &= z\partial_y - y\partial_z, & \varphi_2^u &= yu_z - zu_y, & \varphi_2^v &= yv_z - zv_y \\
 X_3 &= \partial_z, & \varphi_3^u &= u_z, & \varphi_3^v &= v_z \\
 X_4 &= \partial_y, & \varphi_4^u &= u_y, & \varphi_4^v &= v_y \\
 X_5 &= y\partial_y + z\partial_z + u\partial_u + v\partial_v, & \varphi_5^u &= u - yu_y - zu_z, & \varphi_5^v &= v - yv_y - zv_z \\
 X_\alpha &= \alpha(t, x, y, z)\partial_u + \alpha_t(t, x, y, z)\partial_v, & \varphi_\alpha^u &= \alpha, & \varphi_\alpha^v &= \alpha_t \\
 X_\beta &= \beta_z(y, z)\partial_x - \beta_y(y, z)\partial_t, & \varphi_\beta^u &= \beta_y v - \beta_z u_x, & \varphi_\beta^v &= \beta_y(Q - u_{xx}) - \beta_z v_x,
 \end{aligned}
 \tag{5.4}$$

where $\alpha(t, x, y, z)$ is an arbitrary smooth solution of the equations

$$\Delta(\alpha) = 0, \quad \alpha_{tt} + \alpha_{xx} = 0, \quad \alpha_{tz} - \alpha_{xy} = 0, \quad \alpha_{ty} + \alpha_{xz} = 0,
 \tag{5.5}$$

whereas $\beta(y, z)$ satisfies the two-dimensional Laplace equation $\Delta(\beta) = 0$.

We shall find the integrals of motion generating the point symmetries that serve as Hamiltonians of the symmetry flows

$$\begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix},
 \tag{5.6}$$

where the symmetry group parameter τ plays the role of time for the symmetry flow (5.6) and $\mathcal{H} = \int_{-\infty}^{+\infty} H \, dx \, dy \, dz$ is an integral of the motion along the flow (4.6), with the conserved density H , that generates the symmetry with the 2-component characteristic φ_u, φ_v . The second equality in (5.6) is the Hamiltonian form of Noether's theorem that gives a relation between symmetries and integrals.

We choose here the Poisson structure determined by our first Hamiltonian operator J_0 since we know its inverse K given by (4.5), which is used in the inverse Noether theorem

$$\begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = K \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix}
 \tag{5.7}$$

determining conserved densities H corresponding to the known symmetry characteristics φ_u, φ_v .

Using (5.7), we reconstruct the Hamiltonians of the flows (5.6) for all variational point symmetries in (5.4). For the scaling symmetries generated by X_1 and X_5 , Hamiltonians do not exist and so they are not variational symmetries. For the rotational symmetry generated by X_2 , the Hamiltonian is

$$H_2 = v(yu_z - zu_y)\Delta(u) - u_x[2(zu_y + yu_z)u_{yz} + u_y^2 + u_z^2].
 \tag{5.8}$$

For translational symmetries generated by X_3 and X_4 , the corresponding Hamiltonians H_3 and H_4 are

$$\begin{aligned}
 H_3 &= vu_z\Delta(u) + \frac{2}{3}u_x(u_yu_{zz} - u_zu_{yz}) \\
 H_4 &= vu_y\Delta(u) + \frac{2}{3}u_x(u_yu_{yz} - u_zu_{yy}).
 \end{aligned}
 \tag{5.9}$$

For the infinite Lie pseudogroups generated by X_α and X_β , the Hamiltonians are

$$H_\alpha = \alpha v\Delta(u) + \frac{1}{2}\alpha_t(u_y^2 + u_z^2) + \alpha(u_zu_{xy} - u_yu_{xz})
 \tag{5.10}$$

and

$$H_\beta = \left(\frac{\beta_y}{2}v^2 - \beta_zu_xv\right)\Delta(u) - \frac{\beta_y}{2}u_{xx}(u_y^2 + u_z^2) + \frac{1}{2}u_x^2(\beta_{yy}u_y + \beta_{yz}u_z) - \varepsilon\beta_yu.
 \tag{5.11}$$

In particular, the Hamiltonian of time translations $X_{\beta=y} = -\partial_t$, that is $H_{\beta=y} = H_1$, coincides with the Hamiltonian (4.1) of the CMA flow. For translations in x , $X_{\beta=z} = \partial_x$, the Hamiltonian is $H_{\beta=z} = -u_x v \Delta(u)$. For a simple example of the symmetry $X_\alpha, X_{\alpha=z} = z\partial_u$, the Hamiltonian is $H_{\alpha=z} = z v \Delta(u) + u_x u_y$ which coincides with the Hamiltonian H_0 (7.3) from an infinite chain of Hamiltonians for a hierarchy of higher commuting flows in section 7.

All of these Hamiltonians of the symmetry flows are conserved densities of the CMA flow (4.6).

6. Recursion operator

Complex recursion operators for symmetries of the heavenly equations of Plebański were introduced in the papers of Dunajski and Mason [8, 9]. We have used them in our method of partner symmetries for obtaining non-invariant solutions of the complex Monge–Ampère equation [10, 11] and Plebański’s second heavenly equation [11] and, in a 2-component form, for generating multi-Hamiltonian structure of Plebański’s second heavenly equation in [1]. However, for CMA we have an additional condition that the equation and its symmetries are real and hence recursion operators should also be real. Therefore, we shall derive real recursion operators in a 2×2 matrix form.

We start with the symmetry condition for the original 1-component form (2.2) of CMA

$$u_{2\bar{2}}\varphi_{1\bar{1}} + u_{1\bar{1}}\varphi_{2\bar{2}} - u_{2\bar{1}}\varphi_{1\bar{2}} - u_{1\bar{2}}\varphi_{2\bar{1}} = 0 \tag{6.1}$$

which in our 2-component notation $u_t = v, \varphi_t = \psi$ and in the real variables reads

$$a(\psi_t + \varphi_{xx}) + Q\Delta(\varphi) - 2c(\psi_y + \varphi_{xz}) + 2b(\psi_z - \varphi_{xy}) = 0. \tag{6.2}$$

The equation for symmetries (6.2) can be set in a 2-term divergence form

$$(D_t - iD_x)[ia(\psi + i\varphi_x) + (b - ic)(\varphi_y + i\varphi_z)] - (D_y - iD_z)[(b + ic)(\psi + i\varphi_x) - iQ(\varphi_y + i\varphi_z)] = 0 \tag{6.3}$$

that implies local existence of the 2-component potential $(\tilde{\varphi}, \tilde{\psi} = \tilde{\varphi}_t)$ defined by the equations

$$\begin{aligned} \tilde{\psi} - i\tilde{\varphi}_x &= (b + ic)(\psi + i\varphi_x) - iQ(\varphi_y + i\varphi_z) \\ \tilde{\varphi}_y - i\tilde{\varphi}_z &= (b - ic)(\varphi_y + i\varphi_z) + ia(\psi + i\varphi_x). \end{aligned} \tag{6.4}$$

We solve the second equation (6.4) with respect to $\tilde{\varphi}$,

$$\tilde{\varphi} = \Delta^{-1}(D_y + iD_z)[a(i\psi - \varphi_x) + (b - ic)(\varphi_y + i\varphi_z)] \tag{6.5}$$

differentiate this with respect to x and use $\tilde{\varphi}_x$ in the first equation (6.4) to obtain $\tilde{\psi}$,

$$\begin{aligned} \tilde{\psi} &= (b + ic)(\psi + i\varphi_x) + Q(\varphi_z - i\varphi_y) \\ &\quad + \Delta^{-1}(D_y + iD_z)D_x[-a(\psi + i\varphi_x) + (c + ib)(\varphi_y + i\varphi_z)]. \end{aligned} \tag{6.6}$$

In a matrix form, the transformation from (φ, ψ) to $(\tilde{\varphi}, \tilde{\psi})$ reads

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R_c \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{6.7}$$

where the operator R_c is defined by

$$R_c = \begin{pmatrix} \Delta^{-1}D_+[-aD_x + (b - ic)D_+] & i\Delta^{-1}D_+a \\ i\{(b + ic)D_x - QD_+ \\ + \Delta^{-1}D_xD_+[-aD_x + (b - ic)D_+]\} & b + ic - \Delta^{-1}D_xD_+a \end{pmatrix}, \tag{6.8}$$

where we define $D_+ = D_y + iD_z$. As we shall immediately show, taking a real part of the operator (6.8) will give us the first real recursion operator R_1 for symmetries:

$$R_1 = \begin{pmatrix} 0 & 0 \\ QD_z - cD_x & b \end{pmatrix} + \Delta^{-1} \begin{pmatrix} D_y(-aD_x + bD_y + cD_z) + D_z(cD_y - bD_z) & -D_z a \\ D_x[D_y(cD_y - bD_z) + D_z(aD_x - bD_y - cD_z)] & -D_x D_y a \end{pmatrix}, \quad (6.9)$$

where Δ^{-1} means operator multiplication. A straightforward, though cumbersome, calculation shows that the operator R_1 commutes with the operator \mathcal{A} (5.3) of the symmetry condition (5.2) on solutions of equations (4.6) and therefore it is indeed a recursion operator for symmetries of the CMA system. This means that if $(\tilde{\varphi}, \tilde{\psi})$ is obtained by transforming a 2-component symmetry characteristic (φ, ψ) of the system (4.6) by the operator R_1

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R_1 \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad (6.10)$$

then $(\tilde{\varphi}, \tilde{\psi})$ is also a symmetry characteristic of (4.6) and so (6.10) is a real recursion relation for symmetries of the CMA system, the real part of the complex recursion (6.7). If we used (6.7) with the complex recursion operator R_c , the imaginary part of (6.7) would then give the constraint

$$R_2 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (6.11)$$

where R_2 is the imaginary part of R_c , which restricts the set of symmetries (φ, ψ) . To avoid this restriction, we choose (6.10) as the definition of the transformed symmetry $(\tilde{\varphi}, \tilde{\psi})$ and not (6.7).

Moreover, vanishing of the commutator $[R_1, \mathcal{A}]$, computed without using the equations of motion (4.6), reproduces the CMA system (4.6) and hence the operators R_1 and \mathcal{A} form a Lax pair for the 2-component system. Indeed, introducing a short-hand notation $F = u_t - v$, $G = v_t + u_{xx} - Q$, $\Phi = F_{xy} - G_z$ and $\chi = F_{xz} + G_y$, we rewrite the CMA system in the form $F = 0$, $G = 0$, so that $\Phi = 0$ and $\chi = 0$ on its solutions, and the commutator reads

$$[R_1, \mathcal{A}] = \begin{pmatrix} 0 & 0 \\ \frac{1}{a} [Q\Delta(F) - 2(b\Phi + c\chi)] D_z & -\Phi \end{pmatrix} + \Delta^{-1} \times \begin{pmatrix} \Phi(D_z^2 - D_y^2) - 2\chi D_y D_z & D_z \Delta(F) \\ + D_y \Delta(F) D_x - \Delta(F_x) D_y - \Delta(G) D_z, & \\ 2(D_z \chi D_x + D_x \Phi D_y) D_z - D_z \Delta(F) D_x^2 & \\ + \chi_x (D_z^2 - D_y^2) + [2\chi_z - \Delta(G)] D_x D_y & D_x D_y \Delta(F) \\ + \Delta(G_y) D_x - \Delta(G_x) D_y & \end{pmatrix} \quad (6.12)$$

so $[R_1, \mathcal{A}] = 0$ implies $F = 0$, $G = 0$, that is, the CMA system (4.6). This real Lax pair is formed by the recursion operator for symmetries and operator \mathcal{A} of the symmetry condition and so it is a Lax pair of the Olver–Ibragimov–Shabat type [12, 13], which is different from the complex Lax pairs suggested by Mason and Newman [14, 15] and Dunajski and Mason [8, 9] and those that we used in [10, 11] in relation to partner symmetries, even if we set our new Lax pair in a 1-component form. Furthermore, the commutator of the complex recursion operator of Mason–Dunajski with the operator of the symmetry condition, in a 1-component form, reproduces the symmetry condition and not the original equation CMA [10].

We note that if we multiply the symmetry condition (6.2) by i , write it again in a two-term divergence form and introduce the corresponding 2-component potential $(\tilde{\varphi}', \tilde{\psi}')$, then we end up with the second complex operator that differs from (6.8) by the interchange of its real and imaginary parts: the new real part R_2 will coincide with the imaginary part of (6.8) whereas the new imaginary part will be equal to $-R_1$. Now the real part of the transformation from (φ, ψ) to $(\tilde{\varphi}', \tilde{\psi}')$ reads

$$\begin{pmatrix} \tilde{\varphi}' \\ \tilde{\psi}' \end{pmatrix} = R_2 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{6.13}$$

with R_2 defined by

$$\begin{aligned} R_2 = & \begin{pmatrix} 0 & 0 \\ bD_x - QD_y & c \end{pmatrix} \\ & + \Delta^{-1} \begin{pmatrix} D_y(bD_z - cD_y) + D_z(-aD_x + bD_y + cD_z) & D_y a \\ D_x[D_y(-aD_x + bD_y + cD_z) + D_z(cD_y - bD_z)] & -D_x D_z a \end{pmatrix}. \end{aligned} \tag{6.14}$$

Again we check that R_2 commutes with the operator \mathcal{A} (5.3) of the symmetry condition (5.2) on solutions of equations (4.6) and therefore this is the second recursion operator for symmetries of the CMA system. The commutator $[R_2, \mathcal{A}]$ has the form similar to (6.12) and hence vanishing of the commutators $[R_2, \mathcal{A}]$, computed without using the equations of motion (4.6), reproduces the CMA system (4.6). In fact, R_2 can be obtained from R_1 by the discrete transformation

$$y \mapsto z, \quad z \mapsto -y \implies b \mapsto c, \quad c \mapsto -b, \quad \Phi \mapsto \chi, \quad \chi \mapsto -\Phi \tag{6.15}$$

while \mathcal{A} remains invariant under (6.15) and so an explicit form of the commutator $[R_2, \mathcal{A}]$ could be obtained by applying (6.15) to (6.12).

Thus, both operators R_1 and R_2 form together with \mathcal{A} two real Lax pairs of the Olver–Ibragimov–Shabat type for the 2-component system (4.6).

7. Two bi-Hamiltonian representations of the CMA system

By the theorem of Magri, given a Hamiltonian operator J and a recursion operator R , RJ is also a Hamiltonian operator [3]. Thereby, by acting with the recursion operator R_1 on the first Hamiltonian operator J_0 (3.7), we obtain the second Hamiltonian operator

$$\begin{aligned} J_1 = R_1 J_0 = & \Delta^{-1} \begin{pmatrix} D_z & -D_x D_y \\ D_x D_y & D_x^2 D_z \end{pmatrix} \\ & + \begin{pmatrix} 0 & \frac{b}{a} \\ -\frac{b}{a} & \frac{c}{a^2}(bD_y - aD_x) + (D_y b - D_x a) \frac{c}{a^2} + \frac{(c^2 - b^2 + \varepsilon)}{2a^2} D_z + D_z \frac{(c^2 - b^2 + \varepsilon)}{2a^2} \end{pmatrix} \end{aligned} \tag{7.1}$$

that is explicitly skew-symmetric. The proof of the Jacobi identity for J_1 is lengthy but can be somewhat facilitated by using Olver’s criterion in terms of functional multi-vectors [6].

Similarly, acting by the recursion operator R_2 on the Hamiltonian operator J_0 , we obtain a Hamiltonian operator that is another companion for J_0 :

$$J^1 = R_2 J_0 = \Delta^{-1} \begin{pmatrix} D_y & D_x D_z \\ -D_x D_z & D_x^2 D_y \end{pmatrix} + \begin{pmatrix} 0 & -\frac{c}{a} \\ \frac{c}{a} & \frac{b}{a^2}(cD_z - aD_x) + (D_z c - D_x a)\frac{b}{a^2} + \frac{(b^2 - c^2 + \varepsilon)}{2a^2} D_y + D_y \frac{(b^2 - c^2 + \varepsilon)}{2a^2} \end{pmatrix} \quad (7.2)$$

that is also explicitly skew-symmetric. The Jacobi identity for J^1 was also proved by using Olver's criterion in terms of functional multi-vectors [6].

The flow (4.6) can be generated by the Hamiltonian operator J_1 from the Hamiltonian density

$$H_0 = zv\Delta(u) + u_x u_y \quad (7.3)$$

so that CMA in the 2-component form (4.6) admits two Hamiltonian representations

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (7.4)$$

and thus it is a *bi-Hamiltonian system*.

The same flow (4.6) can also be generated by the Hamiltonian operator J^1 from the Hamiltonian density

$$H^0 = yv\Delta(u) - u_x u_z \quad (7.5)$$

which yields another bi-Hamiltonian representation of the CMA system (4.6),

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J^1 \begin{pmatrix} \delta_u H^0 \\ \delta_v H^0 \end{pmatrix}. \quad (7.6)$$

We note that the Hamiltonian operator J^1 and the Hamiltonian density H^0 could be obtained, up to an overall minus sign, by applying the discrete transformation (6.15) to J_1 and H_0 , respectively.

Repeating this procedure n times, we obtain a *multi-Hamiltonian representation* of the CMA system with the Hamiltonian operators $J_n = R_1^n J_0$, $J^n = R_2^n J_0$, $J_m^{n-m} = R_1^m R_2^{n-m} J_0$ ($m = 1, 2, \dots, n - 1$) and the corresponding Hamiltonian densities. This procedure will be considered in more detail in the following section for the operator R_1 . The multi-Hamiltonian structure of the CMA system proves its complete integrability in the sense of Magri and hence the complete integrability of the (anti-)self-dual gravity in four real dimensions with either Euclidean or ultra-hyperbolic signature.

A totally different recursion operator for the (anti-)self-dual gravity in complex Einstein spaces was obtained much earlier by Strachan [16] by using a Legendre transformed version of the first heavenly equation, which was derived by Grant [17]. This recursion operator can be factorized which suggests a bi-Hamiltonian structure of the resulting evolutionary equation, though that was not completely proved. However, the evolutionary equation and the related Hamiltonian structures are expressed in complex variables, with the complex 'time' t in particular, and with a complex unknown. Therefore, the corresponding metric will not correspond to the (anti-)self-dual gravity in real Einstein spaces with the Euclidean signature (+ + +). Furthermore, the Poisson bracket contains the unusual operator ∂_t^{-1} that could be avoided in a 2-component formulation.

8. Infinite hierarchy of higher flows

The operators J_0 and J_1 are compatible Hamiltonian operators, i.e. they form a Poisson pencil. This means that every linear combination $C_0J_0 + C_1J_1$ with constant coefficients C_0 and C_1 satisfies the Jacobi identity. This can be more easily verified by using the Olver criterion in terms of functional multi-vectors though the calculation is still very lengthy. We know from the work of Fuchssteiner and Fokas [18] (see also the survey [19] and references therein) that if a recursion operator has a factorized form, as in our case $R_1 = J_1J_0^{-1} \equiv J_1K$, and the factors J_0 and J_1 are compatible Hamiltonian operators, then R_1 is a hereditary (Nijenhuis) recursion operator, i.e. it generates an Abelian symmetry algebra out of commuting symmetry generators. Moreover, the Hermitian conjugate hereditary recursion operator $R_1^\dagger = J_0^{-1}J_1 = KJ_1$, acting on the vector of variational derivatives of an integral of the flow, yields a vector of variational derivatives of some other integral of this flow. Then (7.4) implies that R_1^\dagger generates the Hamiltonian density H_1 from H_0 :

$$R_1^\dagger \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_0^{-1}J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}, \tag{8.1}$$

where R_1^\dagger is defined by

$$R_1^\dagger = \begin{pmatrix} 0 & D_x c - D_z Q \\ 0 & b \end{pmatrix} + \begin{pmatrix} (-D_x a + D_y b + D_z c)D_y & [(D_z b - D_y c)D_y \\ +(D_y c - D_z b)D_z & +(-D_x a + D_y b + D_z c)D_z]D_x \\ aD_z & aD_x D_y \end{pmatrix} \Delta^{-1}. \tag{8.2}$$

The first higher flow of the hierarchy is generated by J_1 acting on the vector of variational derivatives of H_1

$$\begin{pmatrix} u_{t_1} \\ v_{t_1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}, \tag{8.3}$$

where t_1 is the time variable of the higher flow. This flow is nonlocal and the right-hand side of (8.3) is too lengthy to be presented here explicitly.

Now we could generate the next Hamiltonian H_2 of the hierarchy of commuting flows by applying R_1^\dagger to the vector of variational derivatives of H_1 :

$$R_1^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = KJ_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix}. \tag{8.4}$$

Therefore, the second higher flow in the hierarchy has a bi-Hamiltonian representation

$$\begin{pmatrix} u_{t_2} \\ v_{t_2} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = J_1 R_1^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}, \tag{8.5}$$

where the third Hamiltonian operator J_2 is generated by acting with R_1 on J_1 : $J_1 R_1^\dagger = J_1 K J_1 = R_1 J_1 = J_2$. Acting by J_2 on the variational derivatives of H_0 , we obtain the relations

$$J_2 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_1 R_1^\dagger \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_0 R_1^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix}, \tag{8.6}$$

where we have used that $J_1 = J_0(KJ_1) = J_0 R_1^\dagger$. From (8.6) we obtain three-Hamiltonian representation of the first higher flow

$$\begin{pmatrix} u_{t_1} \\ v_{t_1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix}. \tag{8.7}$$

We could also construct the Hamiltonian H_{-1} such that H_0 is generated from H_{-1} by R_1^\dagger :

$$R_1^\dagger \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = J_0^{-1} J_1 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (8.8)$$

that implies a bi-Hamiltonian representation for the zeroth flow

$$\begin{pmatrix} u_{t_0} \\ v_{t_0} \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix}. \quad (8.9)$$

Further we obtain

$$J_2 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = J_1 R_1^\dagger \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (8.10)$$

and the bi-Hamiltonian representation (7.4) of the original 2-component CMA flow becomes a three-Hamiltonian representation of this flow,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix}. \quad (8.11)$$

We could still continue by applying R_1^\dagger to the vector of variational derivatives of H_2 to generate the next Hamiltonian H_3 ,

$$R_1^\dagger \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = K J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = \begin{pmatrix} \delta_u H_3 \\ \delta_v H_3 \end{pmatrix} \quad (8.12)$$

and obtain a bi-Hamiltonian representation for the next higher flow,

$$\begin{pmatrix} u_{t_3} \\ v_{t_3} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_3 \\ \delta_v H_3 \end{pmatrix} = R_1 J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix}, \quad (8.13)$$

where we have used (8.12) and the relation $J_1 K J_1 = R_1 J_1 = J_2$, and so on.

All these constructions can also be applied to the second recursion operator R_2 .

9. Conclusion

Our starting point was the symplectic and Hamiltonian structure of the complex Monge–Ampère equation, set into a 2-component evolutionary form. We have calculated all point symmetries of the CMA system and also, using the inverse Noether theorem, the Hamiltonians of the flows for all variational symmetries. These Hamiltonians yield conservation laws for the CMA flow. We have found two real 2×2 matrix recursion operators R_1 and R_2 for symmetries that commute with the operator \mathcal{A} of the symmetry condition and hence map any symmetry of the CMA system again into a symmetry. The operators R_1 and R_2 together with \mathcal{A} form two Lax pairs for the 2-component CMA system. Acting on the first Hamiltonian operator by each recursion operator, we obtain two new Hamiltonian operators according to Magri’s theorem [3] and two bi-Hamiltonian representations of the complex Monge–Ampère equation in the 2-component form. Repeating this action, we could generate an infinite number of Hamiltonian operators and hence construct a multi-Hamiltonian representation of the CMA system. We show how to construct an infinite hierarchy of higher commuting flows together with the corresponding infinite chain of their Hamiltonians by using a Hermitian conjugate recursion operator. In particular, we arrive at three-Hamiltonian representations for both the CMA flow and the first higher flow and bi-Hamiltonian representations for the zeroth flow and second higher flow.

The results of this paper prove a complete integrability of the (anti-)self-dual gravity in four real dimensions in the sense of Magri (a multi-Hamiltonian representation).

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